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The trace anomaly: results for spinor fields in six dimensions

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Abstract. We compute the trace anomaly for neutrinos in a Ricci-flat space–time of six dimensions. The integrated stress tensor is also computed which allows the form of the single-loop divergences to be calculated. The stress tensors for scalar, vector and graviton are also quoted.

1. Introduction

In the single-loop approximation of field theory it is always very difficult to get either exact or completely general results.

Some sort of general result can be obtained by considering the Schwinger–DeWitt formalism of single-loop processes. DeWitt (1965, 1975) has applied Schwinger's (1951) proper-time method to the problem of quantising neutrino and scalar fields in a classical background gravitational field. In the neutrino case, for example, DeWitt (1965) iterates the Dirac equation by writing,

$$S(x, x') = -\mathrm{i}(\gamma^{\nu}\nabla_{\nu} - m)G(x, x')\gamma^{-1}$$

so that Dirac's equation,

$$\mathrm{i}\gamma(\gamma^{\mu}\nabla_{\mu} + m)S(x, x') = -\delta(x, x')$$

becomes modified to

$$(\gamma^{\mu}\gamma^{\nu}\nabla_{\mu}\nabla_{\nu} - m^2)G(x, x') = -\gamma\gamma^{-1}\delta(x, x').$$

Use of the equations

$$[\nabla_{\mu}, \nabla_{\nu}]G(x, x') = W_{\mu\nu}G(x, x') = \tfrac{1}{4}R_{\mu\nu\sigma\tau}\gamma^{\sigma}\gamma^{\tau}G(x, x')$$

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}I; \qquad \gamma\gamma^{-1} = I$$

$$\gamma^{\mu}\gamma^{\nu}\gamma^{\sigma}\gamma^{\tau}R_{\mu\nu\sigma\tau} = 2RI$$

leads to the Green function equation,

$$(\Box - m^2 + \tfrac{1}{4}RI)G(x, x') = -I\delta(x, x'). \tag{1}$$

The operator which acts on the Green function in equation (1) can be inverted analogously to Schwinger's treatment. DeWitt puts

$$G = \frac{1}{m^2 - \Box - \frac{1}{4}R} = \mathrm{i}\int_0^{\infty} \mathrm{d}s \, \exp[-\mathrm{i}(m^2 - \Box - \tfrac{1}{4}R)S].$$

Taking matrix elements this becomes

$$G(x, x') = \mathrm{i} \int_0^\infty \mathrm{d}s \exp(-\mathrm{i}m^2 s) \langle x'(s)|x''(0)\rangle \tag{2}$$

where the quantum mechanical propagator satisfies the Schrödinger equation,

$$\mathrm{i}\frac{\partial}{\partial s}\langle x'(s)|x''(0)\rangle = -(\Box + \tfrac{1}{4}R)\langle x'(s)|x''(0)\rangle$$

with the boundary condition,

$$\lim_{s\to 0} \langle x'(s)|x''(0)\rangle = I\delta(x', x'').$$

DeWitt solves this equation by the *ansatz*

$$\langle x'(s)|x''(0)\rangle = \frac{-\mathrm{i}}{16\pi^2 s^2} \exp\left(\frac{\mathrm{i}\sigma}{2s}\right) \sum_{n=0}^{\infty} a_n (\mathrm{i}s)^n \tag{3}$$

and derives the first few a_n. He finds,

$$a_1 = \tfrac{1}{12}IR; \qquad a_0 = I$$

$$a_2 = \tfrac{1}{120}I\Box R + \tfrac{1}{288}IR^2 - \tfrac{1}{180}IR_{\mu\nu}R^{\mu\nu} + \tfrac{1}{180}R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}I + \tfrac{1}{12}W_{ij}W^{ij}.$$

Asymptotic expansions of the form of equation (3) are very useful. They can be of help in the renormalisation of the stress tensor or the effective Lagrangian and can be used in the computation of the trace anomaly.

Although DeWitt was the first to calculate the a_n they have also been calculated by others notably Berger (1966) and also McKean and Singer (1967). Gilkey (1975a, b) has given explicit expressions for a_1, a_2 and a_3 for the asymptotic expansion of any propagator which obeys a second-order differential equation. In particular Gilkey's work can be applied to any spin field.

Van Nieuwenhuizen (1977) has considered the use of a single-loop calculation in six dimensions as a stop-gap 'alternative' to a double-loop calculation in four dimensions until someone brave enough to do the double-loop calculation comes along.

Using this analogy he deduced that since the single-loop divergences of the pure graviton field did not vanish on shell, in six dimensions, they were unlikely to do so for the double-loop calculation in four dimensions.

The calculation was time consuming mainly because he was calculating from first principles using Feynman diagrams. In a recent paper (Critchley 1978) I showed how Gilkey's calculations could be used to get this result out very quickly.

In this paper we will repeat the calculation for neutrinos interacting with a background gravitational field in six dimensions, in order to see if there are any fortuitous cancellations in the single-loop divergences which could conceivably occur in the double-loop calculation in four dimensions. We would predict not since the one-loop calculation in four dimensions is known to be infinite.

However we find that the fermion contribution has the same magnitude but opposite sign to the vector field contribution. It is conceivable therefore that the combined photon plus neutrino fields interacting with the gravitational background field could be finite at the two-loop level in four dimensions.

2. Calculation

In terms of the Green function G, given by equation (1) the trace of the stress tensor for massive neutrinos is given by (in d dimensions),

$$\langle T_\mu{}^\mu \rangle = -\mathrm{i}m^2 \lim_{x\to x'} \mathrm{tr}\, G(x, x'). \tag{4}$$

My renormalisation scheme will be to expand equation (4) in terms of the a_n up to the term which is proportional to $a_{d/2}$, this term will then be subtracted from the (massless) classical result $\langle T_\mu{}^\mu\rangle = 0$. We obtain, in the limit $m \to 0$,

$$\langle T_\mu{}^\mu \rangle_{\text{ren}} = -\mathrm{tr}\, E_{d/2} \qquad \text{with } E_{d/2} = \frac{1}{(4\pi)^{d/2}} a_{d/2}. \tag{5}$$

This is one derivation of the trace anomaly. It agrees where tested, with the result of other regularisation schemes.

The analysis in the next few pages will be slightly more general than is required for just the neutrino anomaly and we will derive useful formulae valid for any spin field.

To ease the calculation I will only consider Ricci-flat space–times, $R_{\mu\nu} = 0$. Gilkey derives the asymptotic expansion for the propagator defined by equation (2), where the corresponding Green function satisfies the equation:

$$(\Box - m^2 + E)G(x, x') = -I\delta(x, x'). \tag{6}$$

The neutrino result is thus a special case with $E = \frac{1}{4}RI = 0$ in a Ricci-flat space–time.

For $R_{\mu\nu} = 0$ Gilkey's (1975b) result for E_3 reduces to

$$\begin{aligned}(E_3) = \tfrac{1}{15120}&(27|\nabla R|^2 - 36|R\Box R| + \tfrac{44}{3}X + \tfrac{80}{3}Y)(I)\\ &+ (\tfrac{1}{45}W_{ij;k}W^{ij;k} + \tfrac{1}{180}W_{ij;}{}^{j}W^{ik;}{}_{k} + \tfrac{1}{30}W_{ij}\Box W^{ij}\\ &- \tfrac{1}{30}W_{ij}W^{jk}W_k{}^i + \tfrac{1}{60}W_{ij}R^{ijkl}W_{kl})\\ &+ (\tfrac{1}{6}E\Box E + \tfrac{1}{12}E_{;i}E_;{}^i + \tfrac{1}{6}E^3 + \tfrac{1}{12}EW_{ij}W^{ij}).\end{aligned} \tag{7}$$

I have dropped a term $\Box\Box E$ since this will be of the form $\Box\Box R = 0$. In equation (7) my definitions are,

$$\begin{aligned} X &= R_{\mu\nu\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}R^{\alpha\beta\mu\nu}; \qquad & |\nabla R|^2 &= R_{\alpha\beta\rho\sigma;\lambda}R^{\alpha\beta\rho\sigma;\lambda}\\ Y &= R_{\mu\nu\rho\sigma}R^{\mu\alpha\rho\beta}R^{\nu}{}_{\alpha}{}^{\sigma}{}_{\beta}; & |R\Box R| &= R_{\alpha\beta\rho\sigma}\Box R^{\alpha\beta\rho\sigma}.\end{aligned} \tag{8}$$

The spin curvature W_{ij} is defined by

$$[\nabla_\mu, \nabla_\nu]G(x, x') = W_{\mu\nu}G(x, x').$$

It is possible to simplify (7) further. The spin curvature satisfies the relation (compare van Nieuwenhuizen 1977),

$$W_{\rho\sigma;\beta\alpha} - W_{\rho\sigma;\alpha\beta} = [W_{\alpha\beta}, W_{\rho\sigma}] - R^\lambda{}_{\rho\alpha\beta}W_{\lambda\sigma} - R^\lambda{}_{\sigma\alpha\beta}W_{\rho\lambda}. \tag{9}$$

Using equation (9) one can show

$$2(W^3) = (W_{\rho\sigma;[\rho,\alpha]}W^{\sigma\alpha}) + R^\lambda{}_{\sigma\alpha\rho}(W^\rho{}_\lambda W^{\sigma\alpha}). \tag{10}$$

The cyclic relation $W_{(\mu\nu;\rho)} = 0$ enables equation (10) to be written

$$(W^3) = \tfrac{1}{2}R^\lambda{}_{\sigma\alpha\rho}(W^\rho{}_\lambda W^{\sigma\alpha}) + \tfrac{1}{4}(W_{\mu\nu}\Box W^{\mu\nu}) - \tfrac{1}{2}(W_{\rho\sigma;}{}^{\rho}{}_{\alpha}W^{\alpha\sigma}). \tag{11}$$

For Ricci-flat space-time $W_{\rho\sigma;\rho}$ is zero. According to Dowker and Dowker (1966) we can write $W_{\rho\sigma}=\frac{1}{4}\mathrm{i}R_{\rho\sigma\alpha\beta}J^{\alpha\beta}$, where $J^{\alpha\beta}$ are the generators of the *d-bein* rotation group SO$(1, d-1)$, so that $W_{\rho\sigma;\rho}=\frac{1}{4}\mathrm{i}R_{\rho\sigma\alpha\beta;}{}^{\rho}J^{\alpha\beta}$. With the Bianchi identity, $R_{\rho(\sigma\alpha\beta)}=0$, this becomes

$$W_{\rho\sigma;}{}^{\rho}=\tfrac{1}{4}\mathrm{i}J^{\alpha\beta}[-R_{\rho\alpha\beta}{}^{\rho}{}_{;\sigma}-R_{\rho\beta}{}^{\rho}{}_{\sigma;\alpha})=0. \qquad (12)$$

Using equation (12) one can derive a relation between $|R\Box R|$ and the invariants X and Y defined in equation (8). For the neutrino case, W_{ij} can be found from DeWitt who has, in our notation

$$\begin{aligned}&[\nabla_\mu,\nabla_\nu]G(x,x')=\tfrac{1}{4}R_{\mu\nu\sigma\tau}\gamma^\sigma\gamma^\tau G(x,x')\\ &W_{ij}=\tfrac{1}{4}R_{ij\sigma\tau}\gamma^\sigma\gamma^\tau.\end{aligned} \qquad (13)$$

Substituting into equation (11) and using the relations, (valid in d dimensions)

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\tau)=2^{d/2}(g^{\mu\nu}g^{\sigma\tau}+g^{\mu\tau}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\tau}$$

$$\begin{aligned}\mathrm{tr}\,\gamma^\alpha\gamma^\beta\gamma^r\gamma^s\gamma^\theta\gamma^\pi\\ =2^{d/2}(&g^{\alpha\beta}g^{rs}g^{\theta\pi}-g^{\alpha r}g^{\beta s}g^{\theta\pi}+g^{\alpha s}g^{\beta r}g^{\theta\pi}-g^{\alpha\theta}g^{rs}g^{\beta\pi}\\ &+g^{\alpha\pi}g^{rs}g^{\beta\theta}g^{\beta\theta}-g^{\alpha\beta}g^{r\theta}g^{s\pi}+g^{\alpha\beta}g^{r\pi}g^{s\theta}+g^{\alpha r}g^{\beta\theta}g^{s\pi}-g^{\alpha r}g^{\beta\pi}g^{s\theta}\\ &-g^{\alpha s}g^{\beta\theta}g^{r\pi}+g^{\alpha s}g^{\beta\pi}g^{r\theta}+g^{\alpha\pi}g^{\beta r}g^{s\theta}-g^{\alpha\pi}g^{\beta s}g^{r\theta}-g^{\alpha\theta}g^{r\beta}g^{s\pi}+g^{\alpha\theta}g^{r\pi}g^{s\beta}).\end{aligned} \qquad (14)$$

I find,

$$-|R\Box R|=X+4Y. \qquad (15)$$

Since equation (15) is a relation between invariants it must be valid in all Ricci-flat space–times.

In deriving the trace of six gamma functions I used the well known relations (Leibbrandt 1975)

$$\begin{aligned}&\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}I; \qquad \mathrm{tr}(I)=2^{d/2}\\ &\mathrm{tr}(\gamma_\mu\gamma_\nu)=2^{d/2}g_{\mu\nu}.\end{aligned}$$

Combining equations (11), (12) and (15) I find

$$\begin{aligned}(E_3)=(I)\tfrac{1}{15120}(27|\nabla R|^2-\tfrac{64}{3}X-\tfrac{352}{3}Y)+(\tfrac{1}{45}W_{ij;k}W^{ij;k}+\tfrac{1}{10}W_{ij}W^{jk}W_k{}^i-\tfrac{1}{60}W_{ij}R^{ijkl}W_{kl})\\ +(\tfrac{1}{6}E\Box E+\tfrac{1}{12}E_{;i}E^{;i}+\tfrac{1}{6}E^3+\tfrac{1}{12}EW_{ij}W^{ij}).\end{aligned} \qquad (16)$$

Equation (16) is valid for any spin field whose Green function obeys the second-order differential equation (6). The trace of E_3 is proportional not only to the trace anomaly but is also proportional to the single-loop divergences in the effective Lagrangian. In general,

$$\int \mathrm{d}^6x\sqrt{-g}\langle T_\mu{}^\mu\rangle_{\mathrm{ren}}=(d-6)\int \mathscr{L}^{(1)}\,\mathrm{d}^6x\propto\int \mathrm{d}^6x\sqrt{-g}\,\mathrm{tr}(E_3). \qquad (17)$$

It is a surprising fact that for every case so far examined only the first term, proportional to tr(I) in equation (24) survives the integration over space–time in equation (17). The second and third terms vanish independently on integrating by parts.

Returning to the specific case of the neutrino field we can put $E = 0$. The second term in brackets in equation (16) can be found from (13) and (14). Substitution of (16) and (5) leads to

$$\langle T_\mu{}^\mu \rangle_{\text{ren}} = -\frac{1}{2(4\pi)^3}\left(-\tfrac{1}{126}|\nabla R|^2 + \tfrac{244}{45360}X + \tfrac{1720}{45360}Y\right). \tag{18}$$

I have divided by two so as to get the result for a single 'neutrino'. Substitution of (18) and (17) and using the relations,

$$\int (Y - \tfrac{1}{2}X)\sqrt{-g}\, \mathrm{d}^6x = 0$$

$$\int |\nabla R|^2 \sqrt{-g}\, \mathrm{d}^6x = \int (X + 4Y)\sqrt{-g}\, \mathrm{d}^6x$$

I find

$$\int \mathscr{L}^{(1)}\, \mathrm{d}^6x = -\frac{4}{15120(4\pi)^3(d-6)} \int X\sqrt{-g}\, \mathrm{d}^6x.$$

In the next section I will summarise the known results for other fields.

3. Summary of six-dimensional results

We can write,

$$\langle T_\mu{}^\mu \rangle_{\text{ren}} = \frac{1}{15120(4\pi)^3}(a|\nabla R|^2 + bX + cY)$$

so that,

$$\int \mathrm{d}^6x \mathscr{L}^{(1)} = \frac{1}{15120(4\pi)^3(d-6)}(3a + b + \tfrac{1}{2}c) \int x\sqrt{-g}\, \mathrm{d}^6x.$$

From the results in this paper and those of Critchley (1978) we can compile table 1.

Table 1

Spin	a	b	c	$3a+b+\frac{1}{2}c$
0	27	$-\frac{64}{3}$	$-\frac{352}{3}$	1
$\frac{1}{2}$	60	$-\frac{122}{3}$	$-\frac{860}{3}$	−4
1	−228	$\frac{500}{3}$	$\frac{3128}{3}$	4
2	2007	−4980	−2064	9

We observe that the single-loop divergences for the neutrino and vector fields cancel when added. It would be interesting to complete this table by considering the spin-$\frac{3}{2}$ field.

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